In a favorable gravity field, the heavy fluid can be delivered to the conical tube by gravity from an attached cylinder in which the separation process is concentrated. Under adverse gravity conditions, however, the separation process alone must be relied upon for supply of heavy fluid to the cone. The numerical results show that for values of $G$ greater than 1.0, the boundary-layer flow reverses near the base of the cone, and the heavy fluid cannot be transported by this mechanism toward the apex. Thus, $G = 1.0$ is the limiting condition for operation of a vortex separator in an adverse axial gravity field.

It should be noted that values of $G$ must be determined by using the appropriate value for $\Omega$ in the heavy fluid. This can be done by equating shear stress at the interface as in Ref. 1, or by equating pressures and assuming slip at the surface of the heavy fluid.

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# Covariance Propagation Via Its Eigenvalues and Eigenvectors

G. J. Bierman*
*Litton Systems Inc., Woodland Hills, Calif.*

## Introduction

In many applications of linear filter theory to navigation analysis, it is important to propagate and update the covariance in an accurate fashion. The square root formulations presented in Refs. 1 and 2 were designed to improve accuracy and maintain non-negativity of the covariance during a Kalman update. One must, however, compute a square root covariance at each update; since the covariance itself is propagated in the interim. A notable exception to this occurs when there is no dynamic noise, i.e., $Q$ of Eq. (1) is zero. In this case, the square root may be propagated by means of the transition matrix.[1]

If during the covariance propagation, the accumulated errors (because of the roundoff and/or the propagation algorithm) cause it to lose its semidefinite character, then the square root updating process may fail. Dyer and McReynolds,[4] introduce orthogonal transformations to circumvent the difficulty (of propagating the covariance) when dynamic noise is present. Their analysis was based upon sequential techniques; but, if the dynamics of the problem are time varying, it may be difficult to apply their results.

Andrews[3] presents an algorithm for continuous propagation of a triangular square root by means of differential equations. The differential equations[3] (5) appears difficult to propagate because of the matrix inversion required at each integration step. Further, his proposed update does not preserve the triangular nature of the square root, so that it is necessary to reinitialize the triangular square root after each update.

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* Engineering Specialist, Systems Analysis Department, Guidance and Control Systems Division.

In this Note, we present a method of propagating the covariance by means of differential equations for its time varying eigenvalues and eigenvectors.† Because the eigenvalues are propagation variables, one can arrange the propagation algorithm so that one is assured that the covariance will not lose its positive semidefinite character. In addition, at an update time, one can apply the square root updating procedures suggested in Refs. 2 and 3, since square roots are expressed quite simply in terms of the eigenvector matrix $\mathbf{X}$ and the eigenvalue matrix $\Lambda$; i.e., if $C = \mathbf{X}\Lambda\mathbf{X}'$, $\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_n)$, then $\mathbf{X}\Lambda^{1/2}Y$ is a square root of $C$ where $\Lambda^{1/2} = \text{Diag}(\lambda_1^{1/2}, \ldots, \lambda_n^{1/2})$ and $Y$ is any unitary matrix, because $C = C^{1/2}(C^{1/2})'$. It is hoped that the time history of the eigenvalues and principal axes will give further insight into the dynamics of the linear analysis.

### Derivation of the Differential Equations Governing the Eigenvectors and Eigenvalues of the Covariance

The covariance matrix we are concerned with can be defined as the solution to

$$\dot{C} = AC + (AC)' + Q, \quad C(O) = C_0 \tag{1}$$

$A$ is an $n \times n$ time varying matrix; $C_0$ and $Q$ are symmetric $n \times n$ matrices; $(\dot{\ })$ represents time derivative; and the $(\,')$ denotes matrix transpose. The time argument is omitted for ease of notation.

As pointed out in Bellantoni and Dodge,[2] one may represent $C$ by

$$C = \mathbf{X}\Lambda\mathbf{X}', \quad \mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n) \tag{2}$$

where $\Lambda$ is a diagonal matrix whose elements are the eigenvalues of $C$, and the columns of $\mathbf{X}$ are the orthonormal eigenvectors corresponding to these eigenvalues. From its definition we have

$$\mathbf{X}\mathbf{X}' = E - \text{identity} \tag{3}$$

and consequently,

$$\dot{\mathbf{X}}\mathbf{X}' + \mathbf{X}\dot{\mathbf{X}}' = 0 \tag{4}$$

Therefore,

$$\dot{\mathbf{X}} = \mathbf{X}\Gamma, \quad \Gamma = -\dot{\mathbf{X}}'\mathbf{X} \tag{5}$$

Equation (5) reflects the fact that the columns of $\dot{\mathbf{X}}$ are linear combinations of the basis unit vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$. Further, applying Eqs. (3) and (4) to Eq. (5) results in

$$\Gamma = -(\mathbf{X}\Gamma)'\mathbf{X} = -\Gamma' \tag{6}$$

i.e., $\Gamma$ is skew-symmetric.

Let us now exploit the relationship involving $\mathbf{X}$, $\Lambda$, and $C$, Eq. (2). Consider the following self-explanatory equations:

$$\begin{aligned}
\Lambda &= \mathbf{X}'C\mathbf{X} \\
\dot{\Lambda} &= \dot{\mathbf{X}}'C\mathbf{X} + \mathbf{X}'\dot{C}\mathbf{X} + \mathbf{X}'C\dot{\mathbf{X}} \\
\dot{\Lambda} &= \Gamma'\mathbf{X}'C\mathbf{X} + \mathbf{X}'\dot{C}\mathbf{X} + \mathbf{X}'C\mathbf{X}\Gamma \\
\dot{\Lambda} &= -\Gamma\Lambda + \mathbf{X}'\dot{C}\mathbf{X} + \Lambda\Gamma
\end{aligned} \tag{7}$$

Because of Eq. (6), the diagonal elements of $\Gamma$ are zero, so that the diagonal terms of Eq. (7) are

$$\dot{\lambda}_i = \mathbf{X}'_i\dot{C}\mathbf{X}_i, \; i = 1, \ldots, n \tag{8}$$

These are the required differential equations for eigenvalue propagation.

In the previous paragraph, the diagonal elements of Eq. (7) were equated and this led to the differential equations for the eigenvalues, Eq. (8). Equating the off-diagonal elements will result in the evaluation of the $\Gamma_{ij}$ which define the differ-

† The derivation is patterned after that given by Kalaba et al. in Ref. 5.

ential equations for the eigenvectors, Eq. (5). Two cases are considered; distinct and nondistinct eigenvalues.

When $\lambda_i \neq \lambda_j$, we evaluate the $i - j$th element of Eq. (7), and obtain

$$\Gamma_{ij} = \mathbf{X}'_i\dot{C}\mathbf{X}_j/(\lambda_j - \lambda_i),\ i \neq j, \lambda_i \neq \lambda_j \tag{9}$$

The case of multiple roots, however, requires some discussion.

Recall that if the eigenvectors $\mathbf{X}_i, \mathbf{X}_{i+1}, \ldots, \mathbf{X}_{i+k}$ correspond to the principal value $\lambda_i$, then they satisfy

$$(C - \lambda_i E)\mathbf{X}_j = 0,\ j = i, i+1, \ldots, i+k \tag{10}$$

Therefore,

$$(C - \lambda_i E)\dot{\mathbf{X}}_j = (\dot{\lambda}_i E - \dot{C})\mathbf{X}_j,\ j = i, \ldots, i+k \tag{11}$$

But, from Eq. (5) we have

$$\dot{\mathbf{X}}_j = \sum_{m=1}^{n} \Gamma_{mj}\mathbf{X}_m \tag{12}$$

and substituting this into Eq. (11) results in

$$\sum_{m=1}^{n} \Gamma_{mj}(\lambda_m - \lambda_i)\mathbf{X}_m = (\dot{\lambda}_i E - \dot{C})\mathbf{X}_j \tag{13}$$

Since the eigenvectors $\mathbf{X}_m$ are orthonormal, we must have

$$\begin{aligned}\Gamma_{mj}(\lambda_m - \lambda_i) &= \dot{\lambda}_i\mathbf{X}'_m\mathbf{X}_j - \mathbf{X}'_m\dot{C}\mathbf{X}_j \\ &= \mathbf{X}'_i\dot{C}\mathbf{X}_i\delta_{mj} - \mathbf{X}'_m\dot{C}\mathbf{X}_j\end{aligned} \tag{14}$$

where $\delta_{mj}$ is the Kronecker delta. Because $\lambda_m = \lambda_i$ for $m = i, i+1, \ldots, i+k$, the left side of Eq. (14) is zero, independent of $\Gamma_{mj}$.‡ Although, we must of necessity have $\Gamma_{jj} = 0$ (because $\mathbf{X}_j \perp \dot{\mathbf{X}}_j$), there seems to be no constraints on the $\Gamma_{mj}$ ($m \neq j$; $m,j = i, i+1, \ldots, i+k$). This shows that any component vector of $\dot{\mathbf{X}}_j$ which lies in the null space of $C - \lambda_i E$ is annihilated, and because of this it seems quite reasonable to require that $\dot{\mathbf{X}}_j$ be orthogonal to this subspace; i.e., $\dot{\mathbf{X}}_j$ is to be orthogonal to the space generated by $\mathbf{X}_i, \mathbf{X}_{i+1}, \ldots, \mathbf{X}_{i+k}$. This convention allows us to write

$$\Gamma_{ij} = 0 \text{ when } \lambda_i = \lambda_j \tag{15}$$

for the case of multiple eigenvalues.

Equations (6, 9, and 14) define $\Gamma$ and Eq. (5) defines the differential equations governing the propagation of the eigenvectors.

### Appendix: Summary of Eigenvalue/Eigenvector Propagation Equations

$$\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n),\ \Lambda = (\lambda_1, \ldots, \lambda_n) \tag{A1}$$

Remark: $C\mathbf{X}_i = \lambda_i\mathbf{X}_i$, $\mathbf{XX}' = E$; if $C_0$ is diagonal, then one may take $\mathbf{X} = E$ and $\Lambda = C_0$.

$$\dot{C} = A\mathbf{X}\Lambda\mathbf{X}' + (A\mathbf{X}\Lambda\mathbf{X}')' + Q \tag{A2}$$

Remark: $\dot{C}$ is not integrated, but is used to evaluate $\dot{\Lambda}$ and $\Gamma$. Thus,

$$\dot{\lambda}_i = \mathbf{X}'_i\dot{C}\mathbf{X}_i \tag{A3}$$

$$\Gamma + \Gamma' = 0 \tag{A4.1}$$

and

$$\Gamma_{ij} = \begin{cases}\mathbf{X}'_i\dot{C}\mathbf{X}'_j/(\lambda_j - \lambda_i) & \text{when } j > i \text{ and } \lambda_j \neq \lambda_i \\ 0 & \text{when } j \geq i \text{ and } \lambda_j = \lambda_i\end{cases} \tag{A4.2}$$

$$\dot{\mathbf{X}} = \mathbf{X}\Gamma \tag{A5}$$

‡ The right side is, of course, zero also.


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# Electron Beam Measurements of Rotational Temperatures with Vibrational Temperatures Greater Than 800°K


WILLIAM W. HUNTER JR.*

*NASA Langley Research Center, Hampton, Va.*


### Nomenclature

| | |
|---|---|
| $B_e, \alpha_e$ | = spectroscopic constants defined in Ref. 7 |
| $c$ | = speed of light |
| $E_{v''_1}$ | = characteristic energy of vibrational energy level $v''_1$ |
| $G(K', T_R)$ | = defined in Ref. 1 |
| $h$ | = Planck's constant |
| $(I_{K',K_2''})_{v',v_2''}$ | = intensity of rotational line emission |
| $I_o$ | = reference intensity |
| $K$ | = rotational quantum number |
| $k$ | = Boltzmann constant |
| $N_{K'}$ | = steady-state number density population of rotational energy state |
| $P_{v',v''_1}$ | = band strength |
| $Q_r$ | = rotational state sum |
| $Q_v$ | = vibrational state sum |
| $T_R$ | = rotational temperature |
| $T_v$ | = vibrational temperature |
| $v$ | = vibrational quantum number |
| $X_1, X_2, X_3$ | = consts |
| $\nu$ | = wave number |

*Superscripts*

| | |
|---|---|
| $(\ )'$ | = quantities for excited electronic state $N_2^+B^2\Sigma_u^+$ |
| $(\ )''$ | = quantities for ground state of neutral nitrogen $N_2X^1\Sigma_g^+$ or ionized nitrogen $N_2^+X^2\Sigma_g^+$ |

*Subscripts*

| | |
|---|---|
| $(\ )_1$ | = ground state of neutral nitrogen $N_2X^1\Sigma_u^+$ |
| $(\ )_2$ | = ground state of ionized nitrogen $N_2^+X^2\Sigma_g^+$ |

THE electron beam technique has been used for measuring the molecular rotational and vibrational temperatures $T_R$ and $T_v$, respectively, of highly expanded low-density flows of air and nitrogen. The original formulation of the theoretical model and experimental verification for $T_R$ and

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* Aero-Space Technologist, Gas Parameters Measurement Section, Instrument Research Division. Member AIAA.